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ABSTRACT 22620

The article discusses the use of artificial anisotropy produced by stresses in an isotropic body. The influence of direction and change of direction and interrelationships of waves are discussed. The solution of light diffusion problems is explained in detail, as well as diffusion-connected changes and problems of axis rotation. Author

The optical aspect of the investigation of stresses by the photoelastic method and, first of all, by the diffuse light method is under consideration.

The optical method of investigating stresses, as is ~~shown~~ known, is based on the use of the artificial anisotropy produced by the stresses occurring in an originally isotropic body. In this case, the main axes of the stress ellipsoid and the dielectric constant ellipsoid in the materials ordinarily used in connection with the mentioned stresses coincide, and

$$\left.\begin{aligned} n_x - n_0 &= C_1\sigma_x + C_2(\sigma_y + \sigma_z) \\ n_y - n_0 &= C_1\sigma_y + C_2(\sigma_x + \sigma_z) \\ n_z - n_0 &= C_1\sigma_z + C_2(\sigma_x + \sigma_y) \end{aligned}\right\} \qquad (1)$$

*Numbers given in the margin indicate the pagination in the original foreign text.

where n_x, n_y and n_z are the refraction indexes corresponding to the main axes of x, y and z; n_0 is the refraction index when there are no stresses; σ_x, σ_y and σ_z are the main stresses, and C_1 and C_2 the optical stress factors (ref. 1).

It would be more correct to use the following equation instead of (1):

$$\left.\begin{array}{c}\varepsilon_x - \varepsilon_0 = C_1'\sigma_x + C_2'(\sigma_y + \sigma_z)\\ \text{etc.}\end{array}\right\}, \quad (2)$$

where $\varepsilon_{x,\,y\text{ and }z}$ are the main values of the dielectric constant tensor. In actual practice, however, it is always

$$\left.\begin{array}{c}\varepsilon_{x_i} - \varepsilon_0 \ll \varepsilon_0 = n_0^2\\ x_i = x, y, z\end{array}\right\}, \quad (3)$$

$$\varepsilon_{x_i} - \varepsilon_0 = n_{x_i}^2 - n_0^2 \cong 2n_0(n_{x_i} - n_0)$$

and, consequently, expressions (1) and (2) are equivalent.

Knowing the values of the C_1 and C_2 constants in (1), measuring $n_{x_i} - n_0$ and determining the direction of the main axes of the optical symmetry, it is possible to find σ_{x_i} and the direction of the main stresses. The object of the stress investigation, therefore, is to find n_{x_i} and the direction of the x_i axes. If the $n_{x_i} - n_0$ differences do not depend on coordinates, the problem can be easily solved by the methods known from crystal optics.

What we are interested in, however, are only the problems in which n_{x_i} changes from point to point. Finding n_{x_i} and the x_i axes in a general nonhomogeneous medium is extremely difficult, and the success of the practical use of the photoelastic methods has so far been limited to simple special cases of the overall problem. The basic special case under exclusive investigation has until now been the two-dimensional problem. One of the main stresses here, such as σ_x for example, has a constant magnitude throughout the body; the direction of the corresponding main axis is also invariable, and the directions of /182 the y and z axes are independent of x, as are the σ_y and σ_z magnitudes[1]. Without reduction of the overall examination, it may be assumed that $\sigma_x = 0$ and therefore agreed that

$$\left.\begin{array}{l} n_y - n_0 = C_1\sigma_y + C_2\sigma_z \\ n_z - n_0 = C_1\sigma_z + C_2\sigma_y \end{array}\right\} \quad (4)$$

If the slight spreads along the x axis (that is, perpendicularly to the plate under load in its plane) n_y and n_z, as well as the directions of the main y and z axis, can be found in exactly the same way as in the case of a homogeneous medium; this is the usual investigation procedure. Even if it is assumed that equation (4) and condition $\sigma_x = 0$ are carefully observed in the case of the plate under investigation, one of the above-made assumptions is automatically approximate in nature. It is precisely to the extent that n_y, z depend on y and z that the light normally falling on the plate (that is, along the x axis) does not spread in it in the same direction. The surface of the wave front in the plate is not plane, and the light rays are not parallel to the x axis because of the optical nonhomogeneity of the medium. The fact that the medium is anisotropic is, as is evident, not essential for an estimation of the magnitude of the appropriate correction. We will therefore discuss the passage of light through a plate of isotropic material whose refraction index n depends on coordinate y (fig. 1).

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[1]Strictly speaking, a problem is also considered two-dimensional when σ_x=0 and is characterized by a changing magnitude; what is important is that σ_x, σ_y and σ_z and the directions of the x, y and z axes remain invariable in some direction common to the entire body. This case, however, is considerably more complicated than the one under consideration.

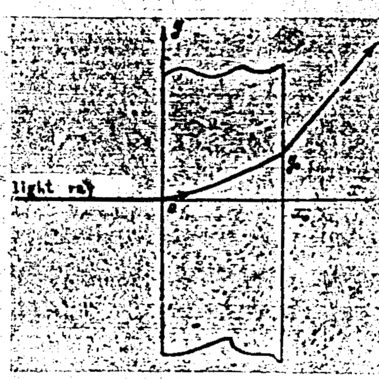

Figure 1

Inasmuch as the refraction index is known to change very slowly at distances equalling a light wave length, the diffusion of light can be well described by an approximation of geometrical optics. The trajectory of the ray shown in fig. 1 can in this case be expressed as follows[1]:

$$\left.\begin{aligned} x &= \int_0^y \frac{dy}{\sqrt{(n(y)/n_0)^2 - 1}} \\ \frac{dy}{dx} &= \sqrt{(n(y)/n_0)^2 - 1} \end{aligned}\right\} \qquad (5)$$

where n_0 is the refraction index, with $y = 0$. It may further be assumed that

$$\sqrt{(n(y)/n_0)^2 - 1} = \sqrt{\frac{2}{n_0}}\,\sqrt{\Delta n(y)},$$

as $n(y) = n_0 + \Delta n(y)$, where $\Delta n(y) \ll n_0$.

[1] In view of the fact the question now under consideration is of no particular interest, we will not go into further details.

Assuming for illustration purposes that $n(y) = \alpha y$, where α is a constant, we have:

$$y = \frac{1}{2n_0}\alpha x^2.$$

If $C = C_1 - C_2 \cong 50$ brewsters, the stresses change at distances of ~1cm per 100 kg/cm^2 $\cong 10^8$ dyne/cm^2 and $\Delta n = \alpha y$, then $\alpha \cong 5 \cdot 10^{-3}$ and $y \cong 2 \cdot 10^{-3} x^2$. Thus, if the plate thickness is $x_0 = 1$ cm, the displacement of the "point of the ray's emergence" amounts to $y_o \cong 2 \cdot 10^{-3}$ cm.

Further, in our example $n_y - n_z \cong C(\sigma_y - \sigma_z) = 5 \cdot 10^{-3}$, $(n_y - n_z)y_o \cong 10^{-5}$, and the change in the delay is

$$\Delta \cong \frac{2\pi}{\lambda}(n_y - n_z)y_0 = \frac{2\pi}{5}$$

that is, relatively fairly large. The above-cited example is apparently indicative of the fact that the distortion of the light ray in the plate is actually not very substantial except for the regions with a large stress gradient.

The possibility of such an effect should be taken into consideration in the case of large stress gradients and when working with materials characterized by a high C value. If the distortion and the related change in the expression for the difference between the lines of two rays, which are inevitably further complicated by a double ray refraction, are appreciable, the advantages provided by the two-dimensional nature of the problem become irrelevant and a simple definition of the stresses impossible.

A fairly simple way of using photoelastic methods to investigate tension by observing the light diffused in the model to be tested (ref. 2) has recently been suggested. We will discuss the problems connected with that method in somewhat greater detail.

Generally, the diffusion of light in a nonhomogeneous anisotropic medium reveals a very complicated picture, as the wave surfaces are not plane (because of the nonhomogeneity of the medium), and the directions of the rays and the normals to the two types of waves capable of propagation in an anisotropic medium do not coincide. Finding the trajectory of the rays and the polarization of both waves with a known n_{x_1} (x,y,z) relationship, much less solving an inverse problem, is practically impossible as it requires the solution of fairly complex differential equations (ref. 3).

In the case of an induced anisotropy, however, the change of the refraction index from point to point, as well as the differences between the main refraction indexes is very small ($\delta n = n_{x_i} - n_{x_k} < 10^{-2}$). That is why the ray distortion is, generally speaking, not very large in a three-dimensional case either. Furthermore, the angle between the normal to the wave front and the direction of the ray is on the order of $\delta n/n_o$ and less than $10^{-2} \simeq 0.5^o$, that is, also very small. It is therefore possible to assume in at least a large number of cases that the light rays are rectilinear, and that the double refraction of the rays in relation to their direction is of minor significance. In a three-dimensional problem, σ_x, σ_y and σ_z and the directions of the main x, y and z axes change along the length of the ray (which we do not distinguish from the normal to the wave). Let the direction of the ray coincide with the x axis, for example; the induction vectors $\vec{D}$ of both independent waves capable of propagating along the x axis will then be directed along the y and z axes. The tension vectors of the electric field $\vec{E}$ of both waves will in this case coincide in the same direction with the $\vec{D}$ vectors. If the direction of the ray (normal) does not coincide with one of the main axes, the $\vec{D}$ and $\vec{E}$ vectors are not parallel; but the perpendicular $\vec{D}$ projection of $\vec{E}$ is approximately $\delta n/n_o$ times smaller than the parallel

$\vec{D}$ projection. We cannot therefore continue to identify the $\vec{D}$ and $\vec{E}$ directions, as in the case of the normal and the ray. The resulting oscillations $\vec{E} = \vec{E}_1\,\vec{E}_2$, where $\vec{E}_1$ and $\vec{E}_2$ are normal oscillations, that is, oscillations corresponding to a definite propagation velocity, are elliptical. The changing phase difference between $\vec{E}_1$ and $\vec{E}_2$ accounts for the changing ellipticity along the ray. /184

Obviously, if it is possible to determine the change of the phase difference between both oscillations along the rays of various directions in the neighborhood of a given point, it is thereby also possible to find n_x, n_y and n_z and the directions of the main x, y and z axes. If the ray is directed along the x axis and we use an approximation of geometrical optics (see below), the mentioned phase difference along the ray will change at distance x by the following magnitude

$$\Delta = \frac{2\pi}{\lambda}(n_y - n_z)x = \frac{2\pi C}{\lambda}(\sigma_y - \sigma_z)x \tag{6}$$

where the mean values on section x should be used for n_y, n_z, σ_y and σ_z, as, for example,

$$n_y = \frac{1}{x}\int_0^x n_y(x)\,dx.$$

Inasmuch as we want to find the magnitude of the tensions at the point, it is clear that the x segment must be small, for otherwise it would be impossible to change from average values to the values at the point; it is the use of light diffusion that makes it possible to determine Δ with a low x value. The nonhomogeneities of the refraction index in a body associated with the density and concentration fluctuation, as well as with dust and various unevenly distributed impurities, result in the diffusion of the light passing through the body; it is precisely because of the diffusion that the path of the ray in the body becomes visible. If the nonhomogeneities are considerably smaller than the light wave and are roughly spherical in shape, the diffusion of light on them is similar to the diffusion on an isotropic oscillator -- the vibrations of such an oscillator, resulting in the appearance of the diffused light, coincide with the direction of the tension vector of the electric field at a given point $\vec{E}$. If a particle (nonhomogeneity) has an elongated shape, it diffuses like an anisotropic oscillator; in this case, the direction of the oscillations does not coincide with $\vec{E}$. Finally, in the case of nonhomogeneities comparable in size to the wavelength, that is, with a larger diameter than $\sim 5 \cdot 10^{-6}$ cm, the nature of the diffusion no longer coincides with that on an oscillator but is considerably more complex.

To avoid additional complications, an attempt should be made to experiment with the first case in which the scattering particles are fairly small and approximately spherical. The diffusion intensity, while the medium is still sufficiently transparent, is proportional to the number of nonhomogeneities (particles). In addition to the usual requirements (ref. 1), the photoelastic materials used in the diffusion method must also satisfy the requirements clearly implied above; the implementation of such requirements (and, first of all, the increase in the number of scattering centers) car make the work a great deal /185
easier.

Let us assume that the diffusion occurs on isotropic oscillators; the object then is to take advantage of the situation and determine, by observing the diffusion, the direction of the $\vec{E}$ vector or, generally speaking, the nature of the oscillations at a given point (the $\vec{E}$ field may be elliptically polarized). This problem can be easily solved if the scattering oscillator is in an isotropic medium. We will make the observation in a plane perpendicular to the primary incident light ray which contains an $\vec{E}$ vector. If the light at a given point is linearly polarized (that is, the $\vec{E}$ direction is invariable), it will not be diffused in the $\vec{E}$ direction, and this will make it possible to determine this direction at once. If the observation is made at θ angle to $\vec{E}$, the intensity of the light will be proportional to $\sin^2\theta$. In the case of an elliptically polarized light, there is no direction in which the diffusion intensity equals zero; however, if the observation is perpendicular to the axes of an ellipse, a maximum diffusion intensity will be observed in one case (if the observation is perpendicular to the large axis), and a minimum intensity in another.

The situation, however, becomes further complicated if we bear in mind that the scattering isotropic oscillator is in an anisotropic medium. The fact that this tension-produced nonhomogeneous anisotropy is not very great makes it possible not to distinguish between the normal and the ray, and to assume that the rays are rectilinear. On the contrary, the difference in the scattering speed of two types of waves running in a given direction must not be disregarded. For this reason, the nature of the polarization of the light diffused within a body will be different upon emerging from the body when in the region of diffusion. The radiation of an oscillator in an anisotropic medium has been previously discussed by the author (ref. 4).

An oscillator vibrating in the direction of $\vec{e}$ will radiate two waves with a different polarization in the direction of $\vec{k}$. If no distinction is made

between $\vec{D}$ and $\vec{E}$, which is quite substantial, the intensity of these waves[1] will be proportional to $\cos^2\theta_1$ and $\cos^2\theta_2$, respectively, where θ_1 and θ_2 are the angles between e and the $\vec{E}_1$ an $\vec{E}_2$ vectors, and $\vec{E}_1$ and $\vec{E}_2$ are the electric vectors in the waves of both types propagating (or rather capable of propagating) in the direction of $\vec{k}$. It follows from this that if the observation is made along $\vec{e}$, the diffusion intensity will be equal to zero, as before. If the vibration produced by the scattering oscillator is elliptical rather than linear, the diffused light emerging from the model will be elliptically polarized even if the observation is carried out along the axes of the ellipse. Only in some particular cases, for example, if the plane of oscillations and the direction of the observation are found in the major tension plane, will the electric vector of the outgoing light be perpendicular to the incident primary ray on the body. It is clear from the above that the most reliable and only universal method of investigation is to fix the points in which the oscillations are /186
linear along the path of the ray.

If the ray is directed along the major x axis (which can be achieved by turning the body), and the direction of the y and z axes along the ray remains unchanged (see below), the diffusion between the points in which the polarization is linear is equal (see (6))

$$d' = \frac{\lambda}{2C(\sigma_y - \sigma_z)}. \qquad (7)$$

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[1]The above-cited expressions apply to a homogeneous anisotropic medium. Therefore, if the medium along the scattered ray changes its characteristics, the intensity correlation of both outgoing components may, in some cases (see below), not be the same as in the region of the diffusion.

A light with linear polarization at point x will also be linearly polarized at x+a but with a different direction of oscillations. Conversely, at a distance of $d = 2d'$ corresponding to the phase difference of $\Delta = 2\pi$, the polarization of the light is exactly the same. An observation made both perpendicular to the ray and in the direction of the oscillations will therefore reveal dark spots along the ray at distances of

$$d = \frac{\lambda}{C(\sigma_y - \sigma_z)}, \tag{7'}$$

from which it is possible to find $\sigma_y - \sigma_z$. In equation (7), C represents the usual relative optical stress factor; in refs. 2 and 5, a constant equalling λ/C is substituted for that magnitude. The d distance is a minimal one if σ_y is the highest of the major stresses and σ_z the lowest. If the ray is not directed along the major axis, the role of σ_y and σ_z is played by the secondary major stresses in the plane perpendicular to the ray.

A change in the directions of the major stresses along the ray tends to complicate the situation, as was pointed out by Drucker and Mindlin (ref. 5). The rotation of the axes may reveal a unique situation in which an approximation of geometrical optics is inapplicable, and which is also a matter of general interest. Let us assume that the light is diffused along the major x axis. All the other characteristics of the medium along that axis, as well as in the other directions, reveal very insignificant changes at wavelength distances (this condition is always realized). In such a case, as it appears at first glance, an approximation of geometrical optics means that both types of waves propagating in a given direction are completely independent of each other, and the changing phase of each of them on the x path is equal to $\Delta = \frac{2\pi}{\lambda}$ ndx,

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where λ is the wavelength in a vacuum, and $n(x)$ the index of refraction corresponding to a given wave. Indeed, such a behavior of both waves should always be observed (ref. 3), if the polarization of these waves in the direction of their propagation does not change. Such behavior is not observable in the case of rotating axes. The rotation of the y and z axes is accompanied by the rotation of the polarization directions of both types of waves. Here, according to the geometrical optics approximation, the electric vector of each of the waves should follow in the direction of the appropriate major axis. This can be shown quite accurately by the use of the general method (ref. 3), but it is also immediately clear. Indeed, we will assume that the $\vec{E}$ vector at point x_1 is directed along the y axis, i.e., that there is only one type of wave (in the second wave the electric vector is directed along the z axis); a geometrical optics /187 approximation shows that one type of wave is not connected with another type (of a different polarization). Only one wave with an electric vector directed against the major y axis should therefore be observed also at point x_2; and the direction of the y axis at x_2 is no longer the same as at x_1. The direction of the oscillations is thus "tied" to the direction of the major axis. At the same time, if the anisotropy is not very pronounced the characteristics (velocities) of both types of waves do not differ a great deal from one another, and a degeneration occurs at the limit point during the onset of isotropy. In an isotropic case, any directions may be selected as y and z axes, and it is clear that the rotation of these directions will not result in the rotation of the plane of the light polarization. Thus in the case of an anisotropic medium, a geometrical optics approximation is inapplicable not only in the presence of large n gradients and the reduction of n to zero (see ref. 3), but also in the extreme case of a mild anisotropy; the inapplicability of geometrical optics in this last case applies only to finding the polarization but not the direction of the light waves.

Let us assume that the direction of the y and z axes rotates evenly, and the angle of their rotation at x distance is[1]

$$\varphi = \frac{A}{2} x. \qquad (8)$$

Since the rotation of the axes is slow, the following condition is automatically realized

$$A\lambda \ll 1. \qquad (9)$$

For example, even if the axes rotate on 2π at a distance of 0.1 cm, $A = 40\pi$ and $A\lambda \approx 6 \cdot 10^{-3}$.

The purpose of using the geometrical optics to find the polarization changes along the ray is to see that

$$R = \frac{A\lambda}{2\pi\delta n} \ll 1. \qquad (10)$$

where δn is the difference between the refraction indexes of both types of waves (that is in the case under consideration: $\delta n = n_y - n_z$), and factor 2π is introduced for convenience. Parameter R is obviously equal to

$$R = \frac{2\varphi}{\Delta} \qquad (11)$$

[see (6) and (8)].

[1]We adhere to the designations adopted in ref. 5.

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In view of the fact that the difference δn in the case of photoelastic investigations can be very small (δn is always less than 0.01), condition (10) does not follow from (9), and the impossibility of its realization is apparently very real.

The problem of light diffusion and, particularly, the diffusion-connected changes in the phase differences between oscillations with various polarizations and arbitrary R value, but assuming the observance of condition (9) and conditions

$$\left(\frac{\partial n}{\partial x}\right)\lambda \ll 1;\ \delta n \ll 1,$$

can be solved in general terms.

This can be done by using the ordinary method of expanding the small parameter λ (ref. 3) by degrees, and taking into account the fact that the off-diagonal components of the tensor of dielectric constants are considerably smaller than the diagonal components. But we will not dwell here on such a /188
consideration as the problem involving the case of rectilinear propagation and its relation to photoelasticity, as well as certain other admissible assumptions, can be strictly and fully resolved. Such a method has, in effect, already been used by Drucker and Mindlin (ref. 5). But it seems to us that their use of designations of the theory of elasticity, and the desire to emphasize a number of details and conclusions, make a repeated brief review of this question expedient.

The equations of light diffusion with a wave normal directed along the x axis in an anisotropic medium, whose properties change only along the x axis, have the following form (ref. 3)

$$\left.\begin{aligned} \frac{d^2E_{y'}}{dx^2} + \frac{\omega^2}{c^2}D_{y'} = 0 \\ \frac{d^2E_{z'}}{dx^2} + \frac{\omega^2}{c^2}D_{z'} = 0 \end{aligned}\right\}, \tag{12}$$

where $E_{y'}$, $E_{z'}$, $D_{y'}$ and $D_{z'}$ are the components of the $\vec{E}$ and $\vec{D}$ vectors on the y' and z' axes which are perpendicular to the x axis, have the same direction everywhere and, consequently, their direction does not coincide with that of the major y and z axes; $w/c = 2\pi/\lambda$, that is w is the angular frequency (the light is assumed to be monochromatic).

The $\vec{D}$ vector is expressed linearly through $\vec{E}$, and in the y and z system of coordinates:

$$\left.\begin{aligned} D_y = (\varepsilon_0 + \delta\varepsilon_y)E_y \\ D_z = (\varepsilon_0 + \delta\varepsilon_z)E_z \end{aligned}\right\}. \tag{13}$$

If at a given point of the axis, y and z form an angle $\varphi(x)$ with the y' and Z' axes, then

$$\left.\begin{aligned} E_y = E_{y'}\cos\varphi + E_{z'}\sin\varphi \\ E_z = -E_{y'}\sin\varphi + E_{z'}\cos\varphi \end{aligned}\right\} \tag{14}$$

and, similarly, for the following connection

D_y and D_z with $D_{y'}$ and $_{z'}$

Accepting correlation (8) and changing the $E_{y'}$, $E_{z'}$, $D_{y'}$ and $D_{z'}$ functions in (12) to E_y, etc., we obtain, as can easily be proved [equations (13) have already been used]

$$\left.\begin{aligned} \frac{d^2E_y}{dx^2} + \left\{\frac{\omega^2}{c^2}(\varepsilon_0 + \delta\varepsilon_y) + \frac{A^2}{4}\right\}E_y - A\frac{dE_z}{dx} = 0 \\ \frac{d^2E_z}{dx^2} + \left\{\frac{\omega^2}{c^2}(\varepsilon_0 + \delta\varepsilon_z) + \frac{A^2}{4}\right\}E_z + A\frac{dE_y}{dx} = 0 \end{aligned}\right\}. \tag{15}$$

The coefficients in equations (15) are constant, and the problem can thus be solved at once. Assuming that $E_{y,z}(x) = E_{y,z}e^{ikx}$, we have

$$\left.\begin{aligned} \left\{-k^2+\frac{\omega^2}{c^2}(\varepsilon_0+\delta\varepsilon_y)+\frac{A^2}{4}\right\}E_y - ikAE_z = 0 \\ ikAE_y + \left\{-k^2+\frac{\omega^2}{c^2}(\varepsilon_0+\delta\varepsilon_y)+\frac{A^2}{4}\right\}E_z = 0 \end{aligned}\right\} \qquad (16)$$

The condition of the existence of a nontrivial solution of that system provides an equation to define k /189

$$\begin{aligned} k^4 - k^2\left\{\frac{\omega^2}{c^2}(2\varepsilon_0+\delta\varepsilon_y+\delta\varepsilon_z)+{}^3/_2A^2\right\}+\frac{\omega^4}{c^4}\left\{\varepsilon_0^2+\varepsilon_0(\delta\varepsilon_y+\delta\varepsilon_z)+\delta\varepsilon_y\delta\varepsilon_z\right\}+ \\ +\frac{A^4}{16}+\frac{\omega^2}{c^2}\frac{A^2}{4}(2\varepsilon_0+\delta\varepsilon_y+\delta\varepsilon_z)=0. \end{aligned} \qquad (17)$$

The solution of equation (17) can be substantially simplified because of the existence of conditions (3) and (9), which are always fulfilled, and also because the $\delta\varepsilon$ and $A\lambda$ magnitudes might be of the same order but a priori $\delta\varepsilon > (A\lambda)^2$. Thus we obtain, correct to a high order of an infinitesimal, (that is, disregarding $(A\lambda)^2$ as compared to $\delta\varepsilon$) for example

$$k^2 = \frac{\omega^2}{c^2}\left\{\varepsilon_0+\frac{\delta\varepsilon_y+\delta\varepsilon_z}{2}\pm\frac{\delta\varepsilon_y-\delta\varepsilon_z}{2}\sqrt{1+\left(\frac{A}{\frac{\omega}{c}\frac{(\delta\varepsilon_y-\delta\varepsilon_z)}{2\sqrt{\varepsilon_0}}}\right)^2}\right\}. \qquad (18)$$

The expression underneath the radical is equal to $1+R^2$, as[1]

$$\frac{A}{\frac{\omega}{c}\frac{\delta\varepsilon_y-\delta\varepsilon_z}{2\sqrt{\varepsilon_0}}}=\frac{A\lambda}{2\pi(n_y-n_z)}=R.$$

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[1] If $\varepsilon_y=\varepsilon_0+\delta\varepsilon_y$, with $\delta\varepsilon_y \ll \varepsilon_0$, then $n_y=\sqrt{\varepsilon_y}\cong\sqrt{\varepsilon_0}+\delta\varepsilon_y/2\sqrt{\varepsilon_0}$.

Further, in view of the smallness of $\delta\epsilon$ and $\delta\epsilon\sqrt{1+R^2}$, we get the final expression

$$k_{\pm}=\frac{\omega}{c}\left\{n_0+\frac{\delta n_y+\delta n_z}{2}\pm\frac{(\delta n_y-\delta n_z)}{2}\sqrt{1+R^2}\right\}, \qquad (19)$$

where $n_0=\sqrt{\epsilon_0},\ \delta n_y=\frac{\delta\epsilon_y}{2\sqrt{\epsilon_0}},\ \delta n_z=\frac{\delta\epsilon_z}{2\sqrt{\epsilon_0}}.$

Thus the waves propagating in the medium in the direction of the x axis are of two types, proportional to $e^{-i(wt - kx)}$ with various $k(k = k\pm)$ values and various types of oscillations. If the axes do not rotate, then

$$k_+=\frac{\omega}{c}(n_0+\delta n_y)\ \&\ k_-=\frac{\omega}{c}(n_0+\delta n_z),$$

and the wave with $E_y \neq 0$ and $E_z = 0$ corresponds to the solution of k_+, and the wave with $E_z \neq 0$ and $E_y = 0$ to the solution of k_-, as it should be. In the event of rotation, the k values are defined by solving (19), and the connection between E_y and E_z in both types of waves can be found by the use of equation (16). The phase difference between both waves in the system of y and z axes amounts to

$$\Delta=\frac{2\pi}{\lambda}(\delta n_y-\delta n_z)\sqrt{1+R^2}\,x. \qquad (20)$$

If condition (10) is observed, this expression changes to (6), that is, the phase difference in the system of y and z axes is the same as in the absence of rotation. In this case there is a full "increase" of the field by rotating axes. The resulting oscillation is linearly polarized at distances equal to (7). The directions of the oscillations in this case vary not only at these distances /190 but also at a distance equal to (7'). This is due to the simultaneous rotation of the axes. If the $\vec{E}$ vector is at first directed along the y or z axis, the oscillation becomes linearly polarized everywhere, and at x distance the direction of the oscillation turns at angle φ [see (8)]. It is clear that the A constant can be easily determined in this case by observing the light diffusion.

If the R parameter is not small, that is it cannot be disregarded, the normal oscillations in the y and z system corresponding to the k and k wave vector are no longer linearly polarized. Namely, as indicated in (15), the connection between E_y and E_z looks like the following[1]

$$E_y = \frac{ikAE_z}{-k^2 + \frac{\omega^2}{c^2}(\varepsilon_0 + \delta\varepsilon_y) + \frac{A^2}{4}} \cong \cong \frac{iA\frac{\lambda}{2\pi}\left(n_0 + \frac{\delta n_y + \delta n_z}{2} \pm \frac{\delta n_y - \delta n_z}{2}\sqrt{1+R^2}\right)}{n_0\{\delta n_y - \delta n_z \mp (\delta n_y - \delta n_z)\sqrt{1+R^2}} E_z, \qquad (21)$$

where the upper symbol applies to the wave with $k = k_+$, and the lower to $k = k_-$. The normal oscillations are thus elliptically polarized. The phase difference between them is defined by correlation (20). The propagation of a wave with any polarization can be ascertained by way of expanding these oscillations into normal ones. It is obvious that the behavior of both waves from the viewpoint of a fixed system of y' and z' coordinates is of greater immediate interest. In that system

$$E_{y'} = E_y \cos\varphi - E_z \sin\varphi, \quad E_{z'} = E_y \sin\varphi + E_z \cos\varphi.$$

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[1]We should point out that the solution applies directly only to a case of even axes rotation; if the rotation is uneven, this solution can be used only in small areas.

and thus in the waves of each type (which differ by the selection of a radical sign) we get, co: rect to an arbitrary amplitude

$$
\left.\begin{aligned}
E_{y'} &= \left\{ i\frac{A\lambda}{2\pi}\left(n_0 + \frac{\delta n_y + \delta n_z}{2} \pm \frac{\delta n_y - \delta n_z}{2}\sqrt{1+R^2}\right)\cos\varphi - \right. \\
&\quad \left. - n_0\left(\delta n_y - \delta n_z \mp (\delta n_y - \delta n_z)\sqrt{1+R^2}\right)\sin\varphi \right\} e^{-i(\omega t - k_\pm x)}, \\
E_{z'} &= \left\{ i\frac{A\lambda}{2\pi}\left(n_0 + \frac{\delta n_y + \delta n_z}{2} \pm \frac{\delta n_y - \delta n_z}{2}\sqrt{1+R^2}\right\}\sin\varphi + \right. \\
&\quad \left. + n_0\left(\delta n_y - \delta n_z \mp (\delta n_y - \delta n_z)\sqrt{1+R^2}\right)\cos\varphi \right\} e^{-i(\omega t - k_\pm x)}
\end{aligned}\right\} \qquad (22)
$$

The actual parts of $E_{y'}$ and $E_{z'}$ are proportional

$$
\left.\begin{aligned}
E_{y'} &\sim \sin\left[\omega t - k_\pm x + \mathrm{arctg}\frac{n_0\left(\delta n_z - \delta n_y \pm (\delta n_y - \delta n_z)\sqrt{1+R^2}\right)\mathrm{tg}\frac{A}{2}x}{\left(n_0 + \frac{\delta n_y + \delta n_z}{2} \pm \frac{\delta n_y - \delta n_z}{2}\sqrt{1+R^2}\right)\frac{A}{2\pi}}\right] \\
E_{z'} &\sim \cos\left[\omega t - k_\pm x + \mathrm{arctg}\frac{\left(n_0 + \frac{\delta n_y + \delta n_z}{2} \pm \frac{\delta n_y + \delta n_z}{2}\sqrt{1+R^2}\right)\frac{A\lambda}{2\pi}\mathrm{tg}\frac{A}{2}x}{n_0\left(\delta n_z - \delta n_y \pm (\delta n_y - \delta n_z)\sqrt{1+R^2}\right)}\right]
\end{aligned}\right\} \qquad (23)
$$

If, for example, $\delta n_y = n_y = 0$, then $(\delta n_y - \delta n_z)\sqrt{1+R^2} = \frac{A\lambda}{2\pi}$ and $k_\pm = \frac{\omega}{c}\left(n_0 \pm \frac{A\lambda}{2\pi}\right)$. /191

$$
\left.\begin{aligned}
E_{y'} &\sim \sin(\omega t - k_\pm x \pm A/2x) = \sin\left(\omega t - \frac{\omega}{c}n_0 x\right) \\
E_{z'} &\sim \cos(\omega t - k_\pm x \pm A/2x) = \cos\left(\omega t - \frac{\omega}{c}n_0 x\right)
\end{aligned}\right\}, \qquad (24)
$$

as $k_\pm = \frac{\omega}{c}\left(n_0 \pm \frac{A\lambda}{2\pi}\right)$

The normal oscillations in this case, as may be seen immediately from (21), have a circular polarization[1]; at the same time, the rotation of the axes cannot

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[1]A complete expression of $E_{y,z}$ would easily reveal that normal oscillations are distinguished by the sign of angular rotation (see, for example (22) assuming that $\delta n_y = \delta n_z = 0$).

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become apparent in the solution in the absence of anisotropy, and it is fictitious in nature, as it follows from (24).

If there is no axis rotation (that is $A = 0$), solution (22) cannot be used, as the division into A has been used above and the already obvious result can be obtained at once from (16).

It is clear from (23) that phase-x relationship of each of the oscillations, as well as the phase difference-x relationship, is fairly complicated. That is why the changes in the intensity-x relationship, occurring during the observation of light diffused in any direction that we can assume to be a z' axis, are not governed by a simple law. That result is quite understandable. Actually, if the resulting oscillation at any point is linearly polarized, then, as it follows from (20), the oscillation will again become linearly polarized for the first time at a distance of

$$d'' = \frac{\lambda}{2(\delta n_y - \delta n_z)\sqrt{1+R^2}} = \frac{\lambda}{2C(\sigma_y - \sigma_z)\sqrt{1+R^2}} \qquad (25)$$

and at a distance of 2d" its polarization in relation to the y and z axes will be the same as at the initial point; generally, $E_{y,z}(x+2d'') = E_{y,z}(x)$.

From the point of view of the y' and z' axes, however, the oscillations at points x and x+2d" differ in their direction as a result of the axis rotation that has occurred in this section; the picture observable in the fixed axes is therefore more complicated. As has already been pointed out, the points with a linear polarization can and should be fixed when the diffusion method is used, and d" determined. Unlike the case when there is no axis rotation, the directions in which the intensity of the diffused light is equal to zero are no longer found in the same plane which is perpendicular to the ray. The bands revealed by an observation in such a plane will not be sharply outlined but somewhat irregular and unclear [see (23)]. In addition to that indication, the rotation will manifest itself primarily in the fact that the distances between the bands and the entire picture do not depend on $\sigma_y - \sigma_z$, but are governed by the law of $\frac{\text{const}}{\sigma_y - \sigma_z}$, as $R = \frac{\text{const}}{\sigma_y - \sigma_z}$. Therefore, if the material is subordinated to correlations (1), while a change in the load fails to implement condition (7), the axes will rotate (barring, of course, any indeterminate complications). The rotation of the axes will undoubtedly complicate the analysis of a three-dimensional state of tension by the photoelastic method; an attempt should therefore be made to keep the effect of the rotation down to a minimum. With the rotation speed fixed at A, the only method in this case is an increase in the load that would lead to an increase of $\delta n_y - \delta n_z$ and a reduction of R. Inasmuch as R is /192 included in the basic expressions of the $\sqrt{1+R^2}$ combination, all we need do is see that R is $<\sim 0.3$, and the effect of the rotation is thus practically entirely eliminated.

A recapitulation of the above justifies the conclusion that, despite a number of complications (double ray refraction of diffused light, the rotation of axes, etc.)[1], the use of diffused light for the solution of three-dimensional photoelastic problems is a highly promising method whose application does not involve too many special requirements.

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[1] In addition to the above-listed requirements, we should note here the complications connected with the desirability of changing the direction of the primary light rays in the body; to avoid refraction on the boundaries, the body should be placed in a liquid with a refraction index of $n_0 = n_0$; in all probability, it would be more convenient to use materials whose n_0 value is as close to a unit as possible.

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